

Noether Symmetries Of A Modified Model In Teleparallel Gravity

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In this paper, we have presented the Noether symmetries of flat FRW spacetime in the context of a new action in Teleparallel Gravity which we construct it based on $f(R)$ version. This modified action contains a coupling between scalar field potential and magnetism. Also, we introduce an innovative approach (B.N.S. Approach) for exact solutions which carry more conserved currents than Noether approach. By data analysis the exact solutions, obtained from Noether approach, late time acceleration and phase crossing are realized, and some deep connections with observational data such as age of universe, the present amount of scale factor, state and deceleration parameters are observed. In B.N.S. approach, we have considered dark energy dominated era.

Keywords: Noether symmetry, Teleparallel gravity, B.N.S. Approach

I. INTRODUCTION

In the last decade, one of the big challenges for physicists is the explanation of the essence and mechanism of the acceleration of our universe [1], in this era of the universe, which has confirmed by some observation data such as supernova type Ia [2] baryon acoustic oscillations [3] weak lensing [4] and large scale structure [5]. However, a plausible elucidation for this is commonly done using the model of a very exotic fluid called dark energy which has negative pressure. Another well-known possibility is to modify Einstein's general relativity [6], making the action of the theory dependent on a function of the curvature scalar R , but at a certain limit of parameters the theory falls in general relativity. This procedure of explaining the accelerated expansion of our universe is known as modified gravity. An alternative, consistently describing the gravitational interaction, is one which only acknowledges the torsion of spacetime, thus cancelling out any effect of the curvature. This approach is known as Teleparallel theory [7] which is demonstrably equivalent to general relativity. Teleparallel gravity enables one to say that gravity is not due to curvature, but the torsion.

The choice of unknown functions, somewhat arbitrary, such as coupling functions and potentials in the equations of motion obtained from the point like lagrangian of the extended models has given rise to objections of fine tuning, the very problem whose solutions have been set out through inflationary theories. Therefore, it is desirable to have a path to derive the potential, or at least some criteria for acceptable potentials. One such approach is based on Noether symmetry and was recently applied by S. Capozziello *et al.* [8], R. de Ritis *et al.* [9], A. K. Sanyal *et al.* [10], and others [11]. Noether approach, representing several conserved currents (Noether currents), is not conducive to any solutions while matching all or a portion of with field equations. The more currents there are the more problems pile up. On the other hand, hidden currents derivable from continuity equation [12], are desired to be included, but when doing so things get worse due to the abundance of currents. The point is, with or without intervening hidden currents, one is compelled to cross out some in order to reach at a solution. In this paper, we introduce a new approach, B.N.S. approach, in section IV which keeps the maximum possible number of conserved currents including Noether, hidden and arbitrary ones.

This paper is organized as follows. In section II we introduce the model and extract the point like lagrangian. In section III we gently present the Noether symmetries, invariants and exact solutions of our model. Moreover, by data analysis, we demonstrate that the observational data corroborate our findings. In section IV we introduce B.N.S. approach and study our model with it, specially in dark energy dominated era. In section V we consider corresponding WDW-equation and eventually in section VI we conclude the results.

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II. THE MODEL

In some articles such as Refs. [13]-[15], authors investigated the $f(R)$ gravitational action as

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{PL}^2}{2} R + \frac{1}{2} \phi_{,\mu} \phi^{,\mu} - V(\phi) - \frac{1}{4} f(\phi)^2 F_{\mu\nu} F^{\mu\nu} \right].$$

These studies led to satisfactory results (inflation, late-time-accelerated expansion, ...). Now, we want to consider the $f(T)$ version of this action completely. For this purpose, the action will simply read

$$S = \int d^4x e \left[\frac{M_{PL}^2}{2} T + \frac{1}{2} \phi_{,\mu} \phi^{,\mu} - V(\phi) - \frac{1}{4} f(\phi)^2 F_{\mu\nu} F^{\mu\nu} \right] \quad (1)$$

where $e = \det(e_\nu^i) = \sqrt{-g}$ with e_ν^i being a vierbein (tetrad) basis, T is the torsion scalar, $\phi_{,\mu}$ stands for the components of the gradient of ϕ and $V(\phi)$ is the scalar field potential. The scalar field as well as strength tensor are constitutive elements of kinetic term. For a given vector field A_μ , we have $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$. It is worth to note that there are several fields such as the vector fields and the nonlinear electromagnetic fields that able to produce the negative pressure effects. In the flat FRW line element,

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2), \quad (2)$$

the scalar torsion takes the form $T = -6\dot{a}^2/a^2$ [17] where a is the scale factor of the universe which depends on time only, and the dot denotes a derivative with respect to time.

Regarding (2), if we introduce the homogeneous and isotropic vector field as

$$A_\mu = (\chi(t); \frac{A(t)}{\sqrt{3}}, \frac{A(t)}{\sqrt{3}}, \frac{A(t)}{\sqrt{3}}), \quad (3)$$

we get

$$F_{\mu\nu} F^{\mu\nu} = \frac{-2\dot{A}^2}{a^2}. \quad (4)$$

The action (1) can be written in the canonical form *i.e.* $S = \int dt L(q, \dot{q}) + \Sigma_0$,

$$L(q, \dot{q}) = -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 + \frac{1}{2}af(\phi)^2\dot{A}^2 - a^3V(\phi), \quad (5)$$

where configuration space is $q = (a, \phi, A)$. We set the reduced Planck mass, M_{PL} , equal to one.

III. NOETHER SYMMETRY

In this section we study Noether symmetry approach for the action (1). We split this section into two subsections. In first subsection III A, we study the general form of Noether symmetry which is perceived as Noether gauge symmetry. However, this terminology is wrong because there is no gauge [15]-[16]. In second subsection III B, we study spatial Noether symmetry which is distinguished as common Noether Symmetry.

A. Noether Symmetry (NS): A General Approach

According to the point like lagrangian (5), the corresponding field equations are as below

$$3\dot{a}^2 + \frac{3a^2\dot{\phi}^2}{2} + \frac{f^2\dot{A}^2}{2} - 3a^2V + 6a\ddot{a} = 0, \quad (6)$$

$$f\dot{A}^2 f' - a^2V' - 3a\dot{\phi}\dot{a} - a^2\ddot{\phi} = 0, \quad (7)$$

$$\dot{a}f\dot{A} + 2af'\dot{A}\dot{\phi} + af\ddot{A} = 0, \quad (8)$$

where the prime indicates the derivative with respect to ϕ . The Hamiltonian constraint or total energy E_L corresponding to (0,0)-Einstein equation becomes

$$\frac{a^2 \dot{\phi}^2}{2} + \frac{f^2 \dot{A}^2}{2} + a^2 V - 3\dot{a}^2 = 0. \quad (9)$$

The pressure and energy density of the scalar field can be written as

$$\begin{aligned} P_\phi &= \frac{\dot{\phi}^2}{2} - V(\phi) + \frac{f^2(\phi) \dot{A}^2}{6a^2}, \\ \rho_\phi &= \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{f^2(\phi) \dot{A}^2}{2a^2}, \end{aligned} \quad (10)$$

Consequently, the EoS parameter for a scalar field could be

$$W_{eff} = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi) + \frac{f^2(\phi) \dot{A}^2}{6a^2}}{\frac{\dot{\phi}^2}{2} + V(\phi) + \frac{f^2(\phi) \dot{A}^2}{2a^2}} \quad (11)$$

To solve the field Eqs. (6) - (8) we use the Noether symmetry approach. A vector field

$$\mathbf{X} = \xi(t, a, \phi, A) \frac{\partial}{\partial t} + \alpha(t, a, \phi, A) \frac{\partial}{\partial a} + \beta(t, a, \phi, A) \frac{\partial}{\partial \phi} + \gamma(t, a, \phi, A) \frac{\partial}{\partial A}, \quad (12)$$

is a Noether symmetry of the Lagrangian (5), if there exists a vector valued function, $G(t, a, \phi, A) \in \Gamma$, where Γ is the space of differential functions such that

$$\mathbf{X}^{[1]} L + L(D_t \xi(t, a, \phi, A)) = D_t G(t, a, \phi, A) \quad (13)$$

in which D_t given by

$$D_t \equiv \frac{\partial}{\partial t} + \dot{a} \frac{\partial}{\partial a} + \dot{\phi} \frac{\partial}{\partial \phi} + \dot{A} \frac{\partial}{\partial A}, \quad (14)$$

is the total derivative operator and $\mathbf{X}^{[1]}$, first-order prolongation is defined by

$$\mathbf{X}^{[1]} = \mathbf{X} + (D_t \alpha - \dot{a} D_t \xi) \frac{\partial}{\partial \dot{a}} + (D_t \beta - \dot{\phi} D_t \xi) \frac{\partial}{\partial \dot{\phi}} + (D_t \gamma - \dot{A} D_t \xi) \frac{\partial}{\partial \dot{A}}. \quad (15)$$

If \mathbf{X} is the Noether symmetry corresponding to the Lagrangian (5), then

$$\mathbf{I} = \xi L + (\alpha - \dot{a} \xi) \frac{\partial L}{\partial \dot{a}} + (\beta - \dot{\phi} \xi) \frac{\partial L}{\partial \dot{\phi}} + (\gamma - \dot{A} \xi) \frac{\partial L}{\partial \dot{A}} - G, \quad (16)$$

is a conserved quantity associated with \mathbf{X} . The Noether symmetry condition for the lagrangian (5) yields the following system of linear partial differential equations

$$\xi_a = \xi_\phi = \xi_A = 0, \quad (17)$$

$$G = -3\alpha a^2 V - a^3 V \xi_t - \beta a^3 V', \quad (18)$$

$$G_a = -6a\alpha_t, \quad G_b = a^3 \beta_t, \quad G_A = a f^2 \gamma_t, \quad (19)$$

$$3\alpha + 2a\beta_\phi - a\xi_t = 0, \quad (20)$$

$$\alpha f + 2\beta a f' + 2a f \gamma_A - a f \xi_t, \quad (21)$$

$$-2\alpha_a + a\xi_t = 0, \quad -6\alpha_\phi + a^2 \beta_a = 0, \quad (22)$$

$$f^2 \gamma_a - 6\alpha_A = 0, \quad f^2 \gamma_a + a^2 \beta_A = 0, \quad (23)$$

1. Zero G

Among the many sets of answers which we found, only $V(\phi)$ and $f(\phi)$ to be non-zero are noteworthy. Our solutions are as below

$$\begin{aligned}\beta &= \frac{-2\sqrt{6}}{3} \left(\frac{c_3 e^{-\sqrt{6}\phi/4}}{a^{3/2}} - c_1 \right), \quad \gamma = \frac{2c_1}{3} A + c_5, \quad \alpha = \frac{-2}{3} \left(\frac{c_3 e^{-\sqrt{6}\phi/4}}{a^{1/2}} \right) + \frac{c_1}{3} a, \\ \xi &= c_1 t + c_2, \quad V(\phi) = V_0 e^{-\sqrt{6}\phi/2}, \quad f(\phi) = f_0 e^{-\sqrt{6}\phi/12}.\end{aligned}\tag{24}$$

Thus, symmetry generators, X_i , on tangent space turn out to be

$$\mathbf{X}_1 = t \frac{\partial}{\partial t} \frac{2\sqrt{6}}{3} + \frac{2A}{3} \frac{\partial}{\partial A} + \frac{a}{3} \frac{\partial}{\partial a}, \quad \mathbf{X}_2 = \frac{\partial}{\partial t}, \quad \mathbf{X}_3 = \frac{-2\sqrt{6}}{3} \frac{e^{-\sqrt{6}\phi/4}}{a^{3/2}} \frac{\partial}{\partial \phi} - \frac{2}{3} \frac{e^{-\sqrt{6}\phi/4}}{\sqrt{a}} \frac{\partial}{\partial a}, \quad \mathbf{X}_4 = \frac{\partial}{\partial A}.\tag{25}$$

So, the corresponding conserved currents are

$$\begin{aligned}\mathbf{I}_1 &= \frac{-ta^3\dot{\phi}^2}{2} + \frac{taf^2\dot{A}^2}{2} - ta^3V + 3ta\dot{a}^2 - 2a^2\dot{a} + \frac{2\sqrt{6}a^3\dot{\phi}}{3} + \frac{2af^2A\dot{A}}{3}, \quad \mathbf{I}_2 = 3a\dot{a}^2 - \frac{a^3\dot{\phi}^2}{2} + \frac{af^2\dot{A}^2}{2} - a^3V, \\ \mathbf{I}_3 &= 4e^{-\sqrt{6}\phi/4}\sqrt{a}\dot{a} - \frac{2\sqrt{6}}{3}e^{-\sqrt{6}\phi/4}a^{3/2}\dot{\phi}, \quad \mathbf{I}_4 = af^2\dot{A}.\end{aligned}\tag{26}$$

Now, we use cyclic variables associated with the Noether symmetry generator \mathbf{X}_3 to simplify the system of equations. The existence of Noether symmetry assures the presence of cyclic variables, say $(t, a, \phi, A) \rightarrow (T, w, u, v)$ such that the Lagrangian becomes cyclic in one of them (w in our case). Cyclic variables can be found by defining a transformation $i : (t, a, \phi, A) \rightarrow (T, w, u, v)$ as an interior product such that $i_{\mathbf{X}_3}dT = 0$, $i_{\mathbf{X}_3}dw = 1$, $i_{\mathbf{X}_3}du = 0$ and $i_{\mathbf{X}_3}dv = 0$ or, put differently, we have the following equations

$$\begin{aligned}\xi \frac{\partial T(t, a, \phi, A)}{\partial t} + \alpha \frac{\partial T(t, a, \phi, A)}{\partial a} + \beta \frac{\partial T(t, a, \phi, A)}{\partial \phi} + \gamma \frac{\partial T(t, a, \phi, A)}{\partial A} &= 0, \\ \xi \frac{\partial w(t, a, \phi, A)}{\partial t} + \alpha \frac{\partial w(t, a, \phi, A)}{\partial a} + \beta \frac{\partial w(t, a, \phi, A)}{\partial \phi} + \gamma \frac{\partial w(t, a, \phi, A)}{\partial A} &= 1, \\ \xi \frac{\partial u(t, a, \phi, A)}{\partial t} + \alpha \frac{\partial u(t, a, \phi, A)}{\partial a} + \beta \frac{\partial u(t, a, \phi, A)}{\partial \phi} + \gamma \frac{\partial u(t, a, \phi, A)}{\partial A} &= 0, \\ \xi \frac{\partial v(t, a, \phi, A)}{\partial t} + \alpha \frac{\partial v(t, a, \phi, A)}{\partial a} + \beta \frac{\partial v(t, a, \phi, A)}{\partial \phi} + \gamma \frac{\partial v(t, a, \phi, A)}{\partial A} &= 0,\end{aligned}\tag{27}$$

where (according to our study *i.e.* \mathbf{X}_3)

$$\alpha = \frac{-2}{3}a^{-1/2}e^{-\sqrt{6}\phi/4}, \quad \beta = \frac{-2\sqrt{6}}{3}a^{-3/2}e^{-\sqrt{6}\phi/4}, \quad \gamma = \xi = 0,\tag{28}$$

i.e. $c_1 = c_2 = c_5 = 0$. The coordinate transformation (27) is not unique and a clever choice can be very advantageous. Moreover, the solution of Eq. (27) is, in general, not defined on the entire space but only locally. Among the many sets of solutions which we found, we choose these solutions

$$T = t, \quad w = \frac{1}{2}a^{3/2}e^{-\sqrt{6}\phi/4}, \quad u = \frac{1}{2}a^{3/2}e^{\sqrt{6}\phi/4}, \quad v = A.\tag{29}$$

Therefore we have $(t, a, \phi, A) \rightarrow (t, \frac{1}{2}a^{3/2}e^{-\sqrt{6}\phi/4}, \frac{1}{2}a^{3/2}e^{\sqrt{6}\phi/4}, A)$ in which w is a cyclic variable. Thus, the scale factor, scalar field, coupling function, and scalar field potential can be written as

$$a = (4uw)^{1/3}, \quad \phi = \frac{2}{\sqrt{6}} \ln\left(\frac{w}{u}\right), \quad f(\phi) = f_0 \left(\frac{u}{w}\right)^{1/6}, \quad V(\phi) = \frac{V_0 u}{w}.\tag{30}$$

The point-like lagrangian (5) in terms of new variables, then reads

$$\mathcal{L} = \mathcal{L}(u, w, A, \dot{u}, \dot{w}, \dot{A}) = \frac{-16}{3}\dot{u}\dot{w} - 4V_0 u^2 + 2^{-1/3}f_0^2 u^{2/3} \dot{A}^2.\tag{31}$$

The Euler Lagrange equations lead to

$$\ddot{u} = 0 \quad , \quad 2^{2/3} f_0^2 \dot{A}^2 - 24V_0 u^{4/3} + 16\ddot{w} u^{1/3} = 0 \quad , \quad 3u\ddot{A} + 2\dot{A}\dot{u} = 0, \quad (32)$$

and the corresponding conserved current (\mathbf{I}_3) will be

$$\tilde{\mathbf{I}}_3 = \frac{16}{3} \dot{u} \quad \rightarrow \quad \ddot{u} = 0. \quad (33)$$

As we observe, this equation doesn't add any new equation. Solutions for the Eq.(32) are

$$\begin{aligned} u(t) &= c_5 t + c_6, \\ A(t) &= c_3 + c_4 \int \frac{dt}{u^{2/3}} = c_3 + \frac{3c_4(c_5 t + c_6)^{1/3}}{c_5} + c_7, \\ w(t) &= \int \left[\int \left(\frac{24V_0 u^{4/3} - 2^{2/3} f_0^2 \dot{A}^2}{16u^{1/3}} \right) dt \right] dt + c_1 t + c_2 \\ &= \frac{1}{32c_5^2} \left[(2^{1/3} 3c_4 f_0)^2 (c_5 t + c_6)^{1/3} + 8V_0 (c_5 t + c_6)^3 \right] + (c_8 + c_1)t + c_2 + c_9, \end{aligned} \quad (34)$$

where $\{c_i \ ; \ i = 1, \dots, 9\}$ are constants of integration. Inserting values of $u(t)$, $A(t)$, and $w(t)$ in Eq.(30), we obtain

$$\begin{aligned} a(t) &= \left[\left(\frac{3f_0 c_4}{2^{7/6} c_5} \right)^2 (c_5 t + c_6)^{4/3} + \frac{V_0}{c_5^2} (c_5 t + c_6)^4 + 4(c_5 t + c_6) \{ (c_1 + c_8)t + c_2 + c_9 \} \right]^{1/3}, \\ \phi(t) &= \frac{\sqrt{6}}{3} \ln \left[\left(\frac{3f_0 c_4}{2^{13/6} c_5} \right)^2 (c_5 t + c_6)^{-2/3} + \frac{V_0}{4c_5^2} (c_5 t + c_6)^2 + (c_5 t + c_6)^{-1} \{ (c_1 + c_8)t + c_2 + c_9 \} \right], \\ f(\phi) &= f_0 \exp \left[\frac{\sqrt{6}\phi}{-12} \right] \\ &\equiv f_0 \left[\frac{32c_5^2 (c_5 t + c_6)}{(2^{1/3} 3c_4 f_0)^2 (c_5 t + c_6)^{1/3} + 8V_0 (c_5 t + c_6)^3 + 32c_5^2 (c_8 + c_1)t + 32c_5^2 (c_2 + c_9)} \right]^{1/6} = f(t), \\ V(\phi) &= V_0 \exp \left[\frac{\sqrt{6}\phi}{-2} \right] \\ &\equiv V_0 \left[\frac{32c_5^2 (c_5 t + c_6)}{(2^{1/3} 3c_4 f_0)^2 (c_5 t + c_6)^{1/3} + 8V_0 (c_5 t + c_6)^3 + 32c_5^2 (c_8 + c_1)t + 32c_5^2 (c_2 + c_9)} \right] = V(t). \end{aligned} \quad (35)$$

To see the behavior of important quantities with these solutions, we choose constants as below

$$\begin{aligned} c_1 &= 0.2795084975 & c_2 &= 1.615579240 \times 10^{10} & c_4 &= 1.120082910 \times 10^{-14} \\ c_5 &= 1.118033986 \times 10^{-21} & f_0 &= \sqrt{-1} = i & V_0 &= 10.00000003 & c_3 &= c_6 = c_7 = c_8 = c_9 = 0. \end{aligned} \quad (36)$$

Perhaps, the amount of f_0 seems strange, but the square of f_0 matters in the action (1), so it is not problematic. We present four figures with data analysis. Figure 1(A) indicates the scale factor with increasing nature expressing first the decelerated and then accelerated expansion of the Universe. Age of universe is $t_0 = 13.80 \text{ Gyr}$ and the present value of the scale factor is $a_0 = 1.00$. Figure 1(B) indicates that the redshift goes down while scale factor increase with time. Present value of the redshift is $z_0 = 0$. Figures 2(A) and 2(B) show the scalar field and Hubble parameter with decreasing natures versus time, as we expect. Present values of the Hubble parameter and scalar field are: $H_0 = 5.7212 \times 10^{-11} \text{ yr}^{-1} \equiv 55.98 \text{ km.s}^{-1} \text{ Mpc}^{-1}$ and $\phi_0 = 39.95$. We do not present plots of the scalar potential $V(\phi)$ and coupling function $f^2(\phi)$ versus ϕ , because their behaviors are obvious (decreasing

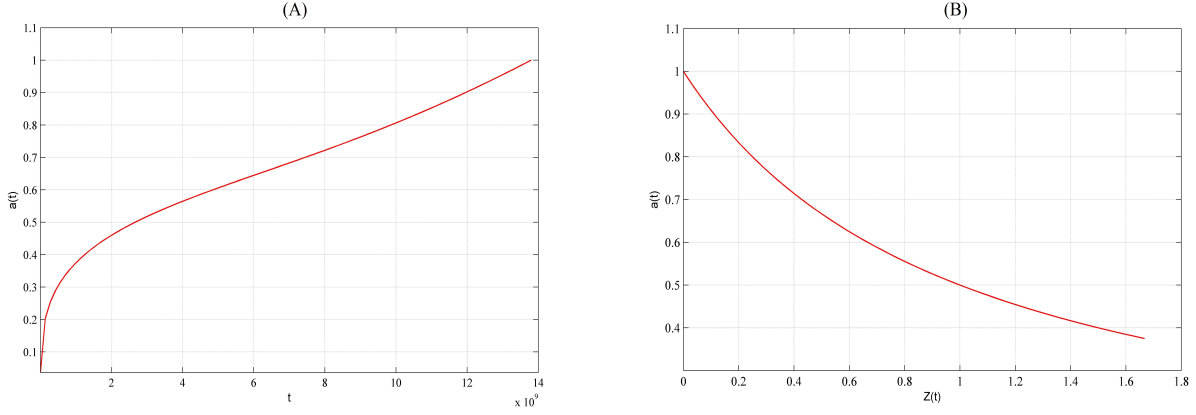


FIG. 1: Plot (A) indicates the scale factor $a(t)$ versus time t at the time range [0.7Myr , 13.8Gyr] while (B) shows the scale factor $a(t)$ versus redshift z at the time range [1Gyr , 13.8Gyr]

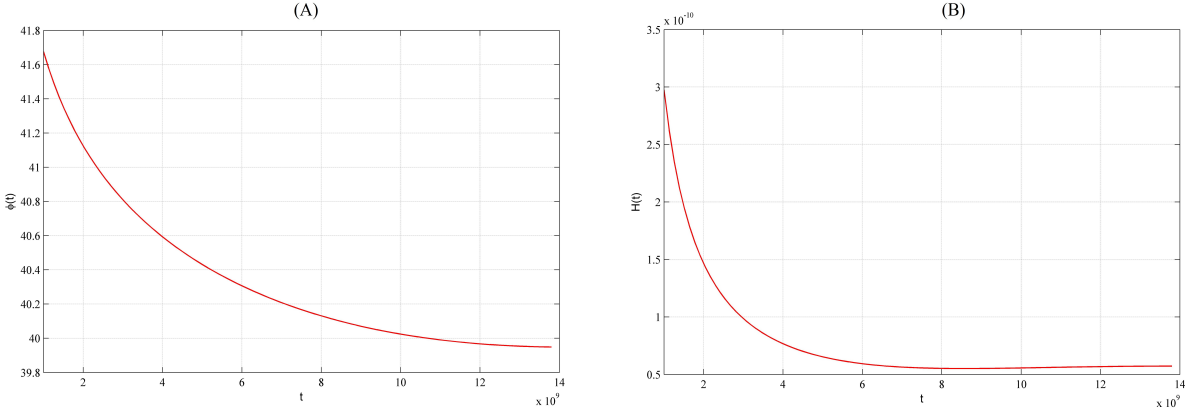


FIG. 2: Plots (A) and (B) indicate the scalar field $\phi(t)$ and Hubble parameter $H(t)$ versus time t at the time range [1Gyr , 13.8Gyr], respectively.

exponentially). In Figure (3) $V(\phi)$ and $f^2(\phi)$ have been plotted with respect to time. Figure 3(A) shows the scalar potential increases with time while Figure 3(B) indicates the coupling function $f^2(\phi)$ decreasing with time, however, the absolute value of $f^2(\phi)$ is increasing such as $V(\phi)$ vs time. Astrophysical data show that W_{eff} lies in a very narrow band close to $W_{eff} = -1$. The behavior of W_{eff} in Figure 4(A) indicates that the crossing of the phantom divide line $W_{eff} = -1$ occurs from the quintessence phase $W_{eff} > -1$ to the phantom phase $W_{eff} < -1$. The present value of the EoS parameter is calculated to be $W_{eff0} = -1.00$. The deceleration parameter, $q = -(a\ddot{a})/\dot{a}^2$, shows first positive, indicative of decelerating universe, and then negative behavior, implying an accelerating universe [Figure. 4 (B)]. So, late-time-accelerated expansion is realized. Present value of the deceleration parameter is measured to be $q_0 = -1.00$, and at the time $t_{ac.} = 6.294Gyr$, we have $q(t_{ac.}) = 0$, so, acceleration starts at the redshift value, $z(t_{ac.}) = 0.524$, which is at about half the age of the universe.

2. Non-Zero G

Because in our study, we focus on the non constant form of $V(\phi)$ and $f(\phi)$, so these would lead to strange results for any choice of function $G(t, a, \phi, A)$ except zero. Therefore, by choosing non-zero G -function, we will not have any conserved current, because all the Noether coefficients are zero.

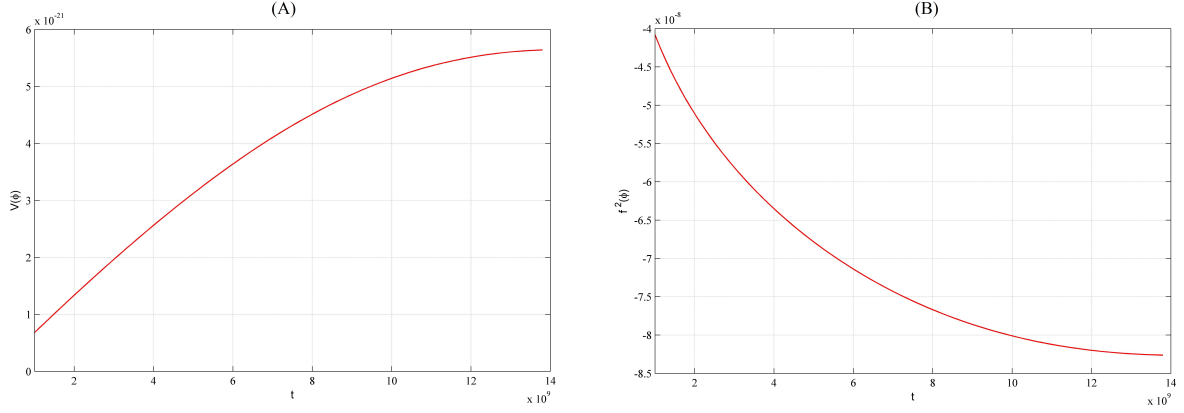


FIG. 3: Plots (A) and (B) indicate the scalar potential $V(\phi)$ and coupling function $f^2(\phi)$ versus time t at the time range [1Gyr , 13.8Gyr] respectively.

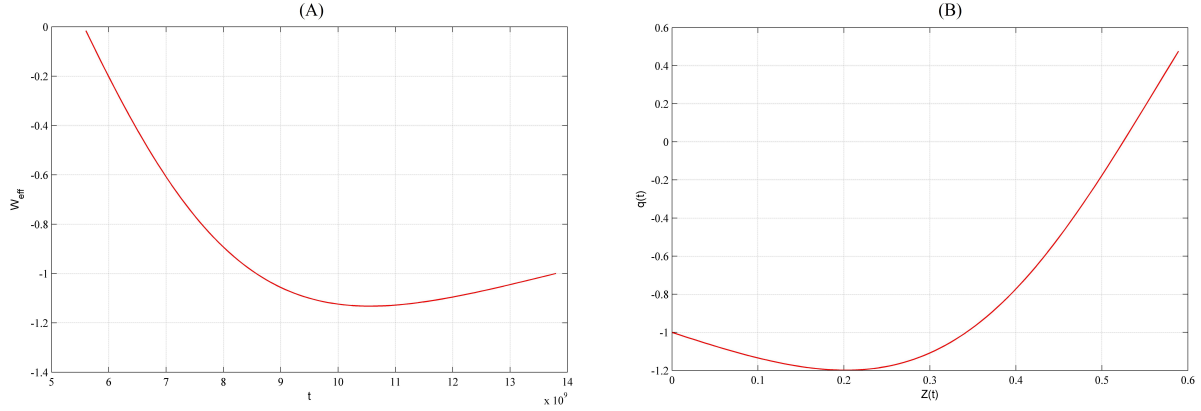


FIG. 4: Plots (A) and (B) indicate the state parameter W_{eff} and deceleration parameter q versus redshift z at the time range [5.6Gyr , 13.8Gyr] respectively.

B. Spatial Noether Symmetry (SNS)

Obviously, for getting the SNS-equations, we must take $G = \xi = 0$. Our solutions for these equations, are as below

$$\beta = \frac{c_1 c_4 \sqrt{6} \left(c_2 e^{\sqrt{6}\phi/4} - c_3 e^{-\sqrt{6}\phi/4} \right)}{a^{3/2}}, \quad \gamma = c_8, \quad \alpha = \frac{c_1 c_4 \left(c_2 e^{\sqrt{6}\phi/4} + c_3 e^{-\sqrt{6}\phi/4} \right)}{a^{1/2}}, \quad (37)$$

$$f(\phi) = c_9 e^{\sqrt{6}\phi/12} \left(c_2 - c_3 e^{-\sqrt{6}\phi/2} \right)^{1/3}, \quad V(\phi) = c_7 \left(c_3^2 e^{-\sqrt{6}\phi/2} - 2c_2 c_3 + c_2^2 e^{\sqrt{6}\phi/2} \right).$$

Symmetry generators, X_i , on tangent space, become

$$\mathbf{X}_1 = \sqrt{\frac{6}{a}} e^{\sqrt{6}\phi/4} \left(\frac{\partial}{\partial a} + \frac{1}{a} \frac{\partial}{\partial \phi} \right), \quad \mathbf{X}_2 = \sqrt{\frac{6}{a}} e^{-\sqrt{6}\phi/4} \left(\frac{\partial}{\partial a} - \frac{1}{a} \frac{\partial}{\partial \phi} \right), \quad \mathbf{X}_3 = \frac{\partial}{\partial A}, \quad (38)$$

Consequently, the corresponding conserved currents take the forms

$$\mathbf{I}_1 = \sqrt{6a} e^{\sqrt{6}\phi/4} \left(-\sqrt{6}\dot{a} + a\dot{\phi} \right), \quad \mathbf{I}_2 = \sqrt{6a} e^{-\sqrt{6}\phi/4} \left(-\sqrt{6}\dot{a} - a\dot{\phi} \right), \quad \mathbf{I}_3 = a f^2 \dot{A}. \quad (39)$$

By a little delicacy, one can find that this case is similar to NS case (*i.e.* the results will be the same). So, it's futile to follow this set of solutions. It's sufficient to say that this generators (and also conserved currents) correspond to \mathbf{X}_3 and \mathbf{X}_4 (so \mathbf{I}_3 and \mathbf{I}_4) in NS approach (See Eqs. (25) and (26)).

IV. BEYOND NOETHER SYMMETRY APPROACH (B.N.S. APPROACH)

In this section, we have innovated an approach for exact solutions. We named it to be "B.N.S. Approach" (B.N.S. is an abbreviation of **B**eyond **N**oether **S**ymmetry). This approach is so useful for extended gravity, because we have some degrees of freedom. Also, we may have more conserved currents. Let us explain this approach.

In any action of extended gravity, we have some unknown functions such as $V(\phi)$ and $f(\phi)$ in our case. As a regular way, we may use Noether approach for defining them. As we observe in our case, and also in almost all other cases in articles, we can't reach the solutions which carry **all** conserved currents or at least more of those. For solving the field equations, we have to remove some of the conserved currents. On the other hand, symmetries have always played a central role in conceptual discussion of the classical and quantum physics. We found that the main problem is the form of these unknown functions, the main culprit in removing some of conserved currents. In the case in which we have new forms of these unknown functions, then the problem can be solved. B.N.S. approach carries out it through a simple way. Suppose that $F_1(\varphi)$, $F_2(\varphi)$, ..., $F_n(\varphi)$ are unknown functions where $\varphi = \varphi(t)$. First of all, we list all field equations and possible conserved currents, then do the maps as follows:

$$\begin{array}{ll}
 1. F_1(\varphi) \rightarrow F_1(t), & \text{So we have: } F'_1(\varphi) \rightarrow \frac{\dot{F}_1(t)}{\dot{\varphi}(t)}, \\
 2. F_2(\varphi) \rightarrow F_2(t), & F'_2(\varphi) \rightarrow \frac{\dot{F}_2(t)}{\dot{\varphi}(t)}, \\
 \vdots & \implies \vdots \\
 n. F_n(\varphi) \rightarrow F_n(t), & F'_n(\varphi) \rightarrow \frac{\dot{F}_n(t)}{\dot{\varphi}(t)},
 \end{array}$$

where the prime indicates a derivative with respect to φ , and the dot indicates differentiation with respect to time. By substituting these in all equations, we may solve our ODE-system easily. After solving the system, we do an inverse map for reaching at the usual form of unknown functions (*i.e.* depending on φ). Perhaps, in some cases, the inverse map be hard to reach at. In such cases, one can do it numerically. In numerical inverse mapping, two options are in order demanding initial values or the time interval, only. Note that one can first carry out Noether approach for getting the conserved currents, and then proceed with this approach *i.e.* D.E-system = {Field equations + Noether Conserved Currents + Other conserved currents} without paying any attention to the form of unknown functions which are obtained by Noether approach. So, one could see that the form of unknown function may be different from those which is derived from Noether approach. Finding the solutions which carry all founded conserved currents with Noether approach is challenging but with B.N.S.-approach this road is paved.

Now, we carry out this approach in our case.

For solving the field equations (Eq.6, 7 and 8), without any loss of generality, we do the maps as follows

$$\left\{ \begin{array}{l} i: f(\phi(t)) \rightarrow f(t), \\ ii: V(\phi(t)) \rightarrow V(t). \end{array} \right. \quad (40)$$

Therefore, we have

$$\left\{ \begin{array}{l} i: f'(\phi) \rightarrow \frac{\dot{f}(t)}{\dot{\phi}(t)}, \\ ii: V'(\phi) \rightarrow \frac{\dot{V}(t)}{\dot{\phi}(t)}. \end{array} \right. \quad (41)$$

Let us, add some other conserved currents from NS (Eq.26). We would like to add $\mathbf{I}_2 = 0$, $\mathbf{I}_3 = 0$ and $\mathbf{I}_4 = c_0^2$, in which c_0 is a constant. Note that Eq. (9) (Hamiltonian constraint) is \mathbf{I}_2 . Substituting Eqs. (40) and (41) in Eqs. (7), (8) and $d\mathbf{I}_4/dt = 0$, we obtain modified system of differential equations

ODE-sys. = {Eq. (6) \cup Eq. (7) \cup Eq. (8) \cup $\mathbf{I}_2 = 0$ \cup $\mathbf{I}_3 = 0$ \cup $d\mathbf{I}_4/dt = 0$ }.

Solving it leads to (Set - 1)

$$\begin{aligned}
\phi(t) &= \text{Any arbitrary function of time.} = b, \\
a(t) &= c_3 \exp\left(\frac{b}{\sqrt{6}}\right), \\
A(t) &= c_1 + c_2 \left(\int \left\{ \exp \left[\int \left(\frac{3\sqrt{6}\dot{b}^3 + 2\sqrt{6}\ddot{b} + 18b\ddot{b}}{6\dot{b}^2 + 2\sqrt{6}\ddot{b}} \right) dt \right] \right\} dt \right), \\
f(t) &= \frac{a \left(-2 \left(3\dot{b}^2 + \sqrt{6}\ddot{b} \right) \right)^{1/2}}{2\dot{A}}, \\
V(t) &= \frac{1}{4} \left(\ddot{b} + 3\dot{b}^2 \right).
\end{aligned} \tag{42}$$

If we do calculations with $\mathbf{I}_4 = c_0^2$, instead of $d\mathbf{I}_4/dt = 0$ the results will be the same. limpidly, we have three symmetry generators (\mathbf{X}_2 , \mathbf{X}_3 and \mathbf{X}_4) with these solutions for the time being. We know that $\phi(t)$ must decay with time. According to Eq. (42), if we set a function with decaying nature for $\phi(t)$, it leads to the same behavior for $a(t)$ as well. So, it is not admissible, hence deserting this solutions. More conserved currents is the reason for this nonphysical solution.

Removing one of conserved currents, $\mathbf{I}_3 = 0$, the results are as below (Set - 2)

$$\begin{aligned}
a(t) &= \text{Any arbitrary function of time.} = F_1 \quad , \quad A(t) = \text{Any arbitrary function of time.} = F_2, \\
f(t) &= c_0 \left(F_1 \dot{F}_2 \right)^{-1/2}, \\
\phi(t) &= c_1 + \int \left[\frac{\sqrt{-6f^2 \dot{F}_2^2 - 18F_1 \ddot{F}_1 + 18\dot{F}_1^2}}{3F_1} \right] dt, \\
V(t) &= \frac{-f^2 \dot{F}_2^2 + 6F_1 \ddot{F}_1 + 12\dot{F}_1^2}{6F_1^2}
\end{aligned} \tag{43}$$

where, $f = f(t)$. As we observe, we have freedom degrees for choosing the form of $a(t)$ in both cases.

• **An example for better understanding the analytical inverse map process.**

In "Set - 1", one can assume following nonphysical form for b

$$b = \sqrt{6}t^2 \quad \rightarrow \quad t^2 = \frac{\phi(t)}{\sqrt{6}}. \tag{44}$$

So, we obtain:

$$a(t) = \exp(t^2) \quad , \quad A(t) = t \exp(3t^2) \quad ,$$

$$f(t) = \frac{\sqrt{-6(6t^2 + 1)} \exp(-2t^2)}{6t^2 + 1} \quad \rightarrow \quad f(\phi) = \frac{\sqrt{-6(\sqrt{6}\phi + 1)} \exp\left(\frac{-2\phi}{\sqrt{6}}\right)}{\sqrt{6}\phi + 1} \quad \rightarrow \quad f(\phi)^2 = \frac{-6 \exp\left(\frac{-2\sqrt{6}\phi}{3}\right)}{\sqrt{6}\phi + 1},$$

$$V(t) = 18t^2 + 1 \quad \rightarrow \quad V(\phi) = \frac{18}{\sqrt{6}}\phi + 3. \tag{45}$$

We proceed with second case. We would like to do it for "Dark energy dominated era". For this purpose, we want to take the form of $\phi(t)$ and $A(t)$ arbitrary. However we have room for choosing the form of $a(t)$, but we want it to arise from the heart of equations spontaneously for comparing with known scale factor for dark energy dominated era (*i.e.* $a_{D.E.} = e^{-1}e^{H_0 t}$. Here, the coefficient H_0 in the exponential, the Hubble constant, is about

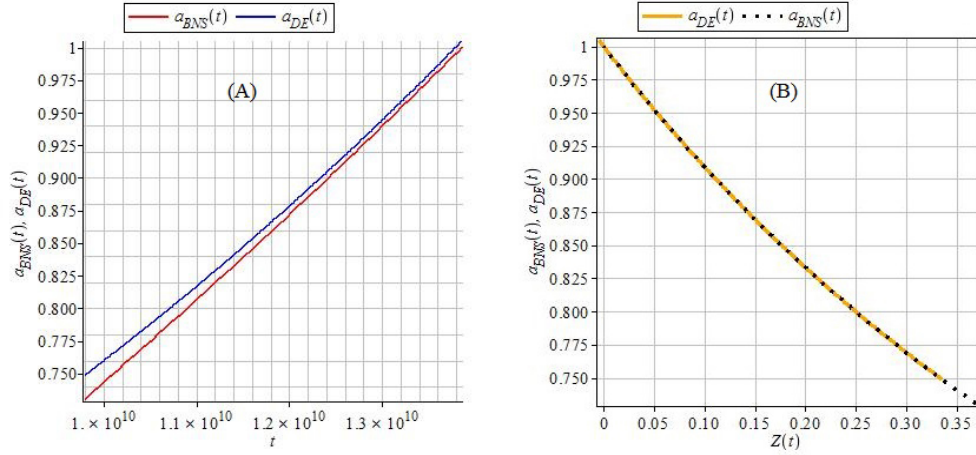


FIG. 5: Plot (A) indicates the scale factors $a_{DE}(t)$ and $a_{B.N.S.}(t)$ versus time t at the time range [9.8Gyr , 13.8Gyr] while (B) shows both scale factors versus their own redshifts z at the same time range evolving.

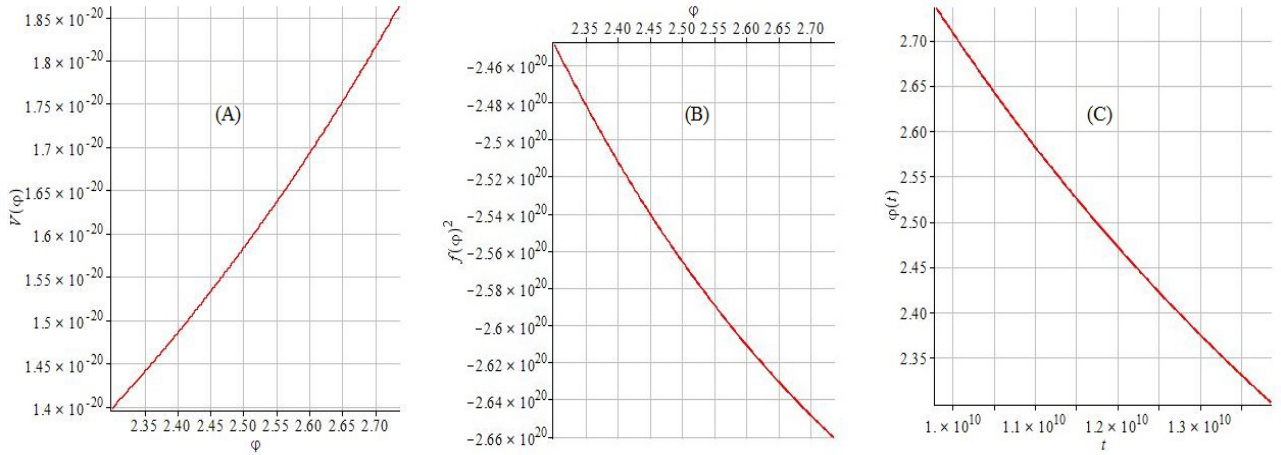


FIG. 6: Plots (A) and (B) indicate the scalar potential $V(\phi)$ and coupling function $f^2(\phi)$ versus scalar field ϕ at the time evolving range [9.8Gyr , 13.8Gyr] respectively while plot (C) shows scalar field ϕ versus time at the same time range.

$7.25 \times 10^{-11} \text{ s}^{-1}$, and the coefficient e^{-1} is for normalizing the scale factor to 1 at present time). We take $\phi(t)$ with decreasing nature function as

$$\phi(t) = \frac{2.708908423 \times 10^5}{\sqrt{t}}, \quad (46)$$

and the form of $A(t)$ as obtained from NS-results (Eq.(34))

$$A(t) = -7.068617997 \times 10^{-14} t^{1/3}, \quad (47)$$

Solving Eq. (43) with Eq. (46) and (47) numerically as a boundary value problem in the dark energy dominated era's time range, [9.8 Gyr, 13.8 Gyr] with these selections

$$c_0 = 1, \quad c_1 = 0, \quad a(9.8 \times 10^9) = 0.7304711450, \quad a(13.8 \times 10^9) = 1, \quad (48)$$

shows deep compatible results with observational data. We present Figures (5) and (6) to demonstrate the results obtained. Figure (5) (A) indicates the behaviors of obtained scale factor, $a_{B.N.S.}(t)$, and known scale factor for dark energy dominated era, $a_{D.E.}(t)$ versus time. It shows a good agreement between $a_{B.N.S.}(t)$ and $a_{D.E.}(t)$. However, maybe, it is not a well comparison, for we studied those versus time. For this reason, we study $a_{B.N.S.}(t)$ and $a_{D.E.}(t)$ versus their own redshifts. Plots overlapping in Figure 5 (B) shows a perfect agreement between $a_{B.N.S.}(t)$ and $a_{D.E.}(t)$ versus $z_{B.N.S.}$ and $z_{D.E.}$ respectively. Figure (6) (C) shows detractive behavior of scalar

field ϕ versus time which leads to decreasing $V(\phi)$ and incremental $f^2(\phi)$ versus ϕ in Figures 6 (A) and (B) respectively. Here, $f(\phi)$ is pure imaginary again, and as mentioned above, it's not problematic. As an example, one can take $a(t) = c_1 \sinh(c_2 t)^{2/3}$, and describe the elaborations of cosmic evolution from matter dominated era till now.

V. WDW-EQUATION

Let us proceed with Eq.(31). So, we have following Hamiltonian

$$\mathcal{H} = -\frac{3}{16}\Pi_u\Pi_w + 4V_0u^2 + 2f_0^{-2}u^{-2/3}\Pi_A^2, \quad (49)$$

where, $\{\Pi_j = \partial\mathcal{L}/\partial\dot{Q}^j ; Q^j \in \{w, u, v = A\}\}$ are conjugated momenta of configuration space. By a straightforward canonical quantization procedure, we have

$$\begin{aligned} \Pi_j &\rightarrow \hat{\Pi}_j = -i\partial_j, \\ \mathcal{H} &\rightarrow \hat{\mathcal{H}}(Q^j, -i\partial_{Q^j}). \end{aligned} \quad (50)$$

Hamiltonian constraint gives the Wheeler-De Witt equation

$$\left[-\frac{3}{16}(-i\partial_u)(-i\partial_w) + 4V_0u^2 + 2f_0^{-2}u^{-2/3}(-i\partial_A)^2 \right] |\Psi(w, u, A)\rangle = 0, \quad (51)$$

in which $|\Psi(w, u, A)\rangle$ is the wave function of the universe. Pursuant to Noether symmetry, if we use the following two conserved currents

$$\Pi_w = \Sigma_1, \quad \Pi_A = \Sigma_2, \quad (52)$$

and according to

$$|\Psi\rangle = \sum_{j=1}^m e^{i\Sigma_j Q^j} |\chi(Q^l)\rangle, \quad m < l \leq n, \quad (53)$$

where m is the number of symmetries, l are the directions where symmetries do not exist, n is the total dimension of minisuperspace, we have

$$|\Psi(w, u, A)\rangle = e^{i\Sigma_1 w} e^{i\Sigma_2 A} |\Theta(u)\rangle \quad (54)$$

After putting this solution in Eq. (51) and solving it, we get this perfect solution

$$|\Psi(w, u, A)\rangle = c_1 e^{i\Sigma_1 w} e^{i\Sigma_2 A} e^{\frac{b_2 u^3 + 9b_3 u^{1/3}}{-3b_1}} \quad (55)$$

where $b_1 = 3i\Sigma_0/16$, $b_2 = 4V_0$, $b_3 = 2f_0^{-2}u^{-2/3}\Sigma_1^2$, and c_1 is an integration constant. It is clear that oscillating feature of wave function of universe, recovers the so-called Hartle criterion [18].

VI. CONCLUSION

In this paper, we studied an original action in Teleparallel gravity which has never been introduced in the literatures. However the $f(R)$ version of the action (1) has been introduced and studied in some papers [13]-[15]. By the use of Noether approach, we found that late-time-accelerated expansion is realized with this model. Our data analysis showed that the age of universe is 13.86 *Gyr*, the present amount of scale factor, deceleration and EoS parameters are $a_0 = 1.00$, $q_0 = -1$ and $W_{eff} = -1$, respectively and the scalar field, ϕ , and coupling function, $f^2(\phi)$, with decreasing nature while scalar potential, $V(\phi)$, increasing with time. Considering deceleration parameter, we learned that the universe starts accelerating from $z = 0.524$, which is equivalent to $t_{ac.} = 6.294 \text{ Gyr}$. The resulting model crosses the phantom divide line from the quintessence phase to the phantom phase. We obtained the quadratic coupling function, $f^2(\phi)$, to be real in the action (1) but $f(\phi)$ itself was pure imaginary. We presented an original approach as "B.N.S. Approach" for easy solving the field equations with keeping almost all conserved currents which we want to have, and showed that unlike Noether approach, we have solutions with

this approach. Our solutions could describe dark energy dominated era. And in the section V we considered WDW-equation and showed that wave function of universe has oscillating features which in the cosmic evolution recovers the so-called Hartle criterion.

Finally, we would like to compare our NS-results with SNS-results of Ref. [14] which considered $f(R)$ version of the action (1). By taking the incorrect vector field as $A_\mu = (0; 0, 0, A(t))$, they showed that the scale factor is about $a(t) \sim (t^4 + t^{4/3} + t)^{1/3}$ while in our case it is about $a(t) \sim (t^4 + t^{4/3} + t + t^2)^{1/3}$. According to the extra term, t^2 , in the parenthesis, other things are different. However, regarding the FLRW spacetime, we took different vector field as (3). Anyway, both versions, show late-time-accelerated expansion. It is worth to note that the behavior of scalar field is different; that is, in our case it has decreasing nature while in that paper it has increasing nature after a little detractive behavior. In that paper, they studied qualitative behavior of $a(t)$ and $\phi(t)$ versus time only, so we can't discuss more. Howbeit, I am sure that in that paper, one can show pure imaginary nature of $f(\phi)$ by data analysis, such as our case. However we have $f^2(\phi)$ in that action, so it is not problematic.

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